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On the spinor representation of surfaces in Euclidean 3-space ★

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Abstract

The aim of the present paper is to clarify the relationship between immersions of surfaces and solutions of the Dirac equation. The main idea leading to the description of a surface M^2 by a spinor field is the observation that the restriction to M^2 of any parallel spinor ψ on $\mathbb{R}^3$ is a non-trivial spinor field on M^2 of constant length which is a solution of the inhomogeneous Dirac equation. Vice versa, any solution of the equation $D(\psi) = H \cdot \psi$ of constant length defines a symmetric endomorphism satisfying the Gauss- and Codazzi equations, i.e. an isometric immersion of M^2 into the 3-dimensional Euclidean space.

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1. Introduction

The Weierstraß formula describes a conformal minimal immersion of a Riemann surface M^2 into the 3-dimensional Euclidean space $\mathbb{R}^3$. It expresses the immersion in terms of a holomorphic function g and a holomorphic 1-form μ as the integral

$$f = \mathrm{Re}\left(\int (1 - g^2, \mathrm{i}(1 + g^2), 2g)\mu\right) : M^2 \to \mathbb{R}^3.$$

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On the other hand let us consider the spinor bundle S over M^2. This 2-dimensional vector bundle splits into

$$S = S^+ \oplus S^- = \Lambda^0 \oplus \Lambda^{1,0}.$$

Therefore the pair (g, μ) can be considered as a spinor field φ on the Riemann surface. The Cauchy–Riemann equation for g and μ is equivalent to the Dirac equation

$$D(\varphi) = 0.$$

The choice of the Riemannian metric in the fixed conformal class of M^2 is not essential since the kernel of the Dirac operator is a conformal invariant.

A similar description for *an arbitrary* surface $M^2 \hookrightarrow \mathbb{R}^3$ is possible and has been pointed out probably for the first time by Eisenhardt (1909). This representation of any surface in $\mathbb{R}^3$ by a spinor field φ on M^2 satisfying the Dirac equation

$$D(\varphi) = H\varphi \tag{$*$}$$

involving the mean curvature H of the surface has been used again in some recent papers (see [5,8,10–12]). However, the mentioned authors describe the relationship between surfaces in $\mathbb{R}^3$ and solutions of Eq. $(*)$ in local terms in order to get explicit formulas. The aim of the present paper is to clarify the mentioned representation of surfaces in $\mathbb{R}^3$ by solutions of the equation $D(\varphi) = H\varphi$ in a *geometrically invariant way*. It turns out that the main idea leading to the description of a surface by a spinor field φ is simple: Consider an immersion $M^2 \hookrightarrow \mathbb{R}^3$ and fix a parallel spinor Φ on $\mathbb{R}^3$. Then the restriction $\varphi = \Phi_{|M^2}$ of Φ to the surface is (with respect to the inner geometry of M^2) a non-trivial spinor field on M^2 and defines a spinor φ^* of constant length which is a solution of the inhomogeneous Dirac equation

$$D(\varphi^*) = H\varphi^*.$$

Conversely, given a solution φ of Eq. $(*)$ with constant length there exists a symmetric endomorphism $E : T(M^2) \to T(M^2)$ such that the spinor field satisfies a "twistor type equation"

$$\nabla_X \varphi = E(X) \cdot \varphi.$$

The resulting integrability conditions for the endomorphism E are exactly the Gauß and Codazzi equations. As a consequence, the solution φ of the Dirac equation

$$D(\varphi) = H\varphi, \quad |\varphi| \equiv \text{const} > 0$$

yields an isometric immersion of M^2 into $\mathbb{R}^3$. In a similar way one obtains the description of conformal immersions using the well-known formula for the transformation of the Dirac operator under a conformal change of the metric (see [2]).

2. The Dirac operator of a surface immersed into a Riemannian 3-manifold

Let Y^3 be a 3-dimensional oriented Riemannian manifold with a fixed spin structure and denote by M^2 an oriented surface isometrically immersed into Y^3. Because the normal bundle of M^2 is trivial, the spin structure of Y^3 induces a spin structure on the Riemannian surface M^2. The spinor bundle S of the 3-manifold Y^3 yields by restriction the spinor bundle of the surface M^2. Over M^2 this bundle decomposes into

$$S = S^+ \oplus S^-$$

where the subbundles $S^\pm$ are defined by (see [3])

$$S^\pm = \{\varphi \in S \colon \mathrm{i} \cdot e_1 \cdot e_2 \cdot \varphi = \pm\varphi\}.$$

Here $\{e_1, e_2\}$ denotes an oriented orthonormal frame in $T(M^2)$ and $X \cdot \varphi$ means the Clifford multiplication of a spinor $\varphi \in S$ by a vector $X \in T(M^2)$. Since in the 3-dimensional spin representation the relation

$$e_1 \cdot e_2 = e_3$$

holds, we can replace the Clifford product $e_1 \cdot e_2$ by the normal vector $\mathbf{N}$ of $M^2 \hookrightarrow Y^3$:

$$S^\pm = \{\varphi \in S : \mathrm{i} \cdot \mathbf{N} \cdot \varphi = \pm\varphi\}.$$

Consider a spinor field Φ defined on the 3-manifold Y^3. Its restriction $\varphi = \Phi_{|M^2}$ is a spinor field defined on M^2 and decomposes therefore into $\varphi = \varphi^+ + \varphi^-$ with

$$\varphi^+ = \tfrac{1}{2}(\varphi + \mathrm{i}\mathbf{N} \cdot \varphi), \quad \varphi^- = \tfrac{1}{2}(\varphi - \mathrm{i}\mathbf{N} \cdot \varphi).$$

We denote by ∇^{Y^3} and ∇^{M^2} the covariant derivatives in the spinor bundles on Y^3 and M^2 respectively. For any vector $X \in T(M^2)$ we have the well-known formula (see [2])

$$\nabla^{Y^3}_X(\Phi) = \nabla^{M^2}_X(\varphi) - \tfrac{1}{2}(\nabla_X \mathbf{N}) \cdot \mathbf{N} \cdot \varphi.$$

The endomorphism $X \to (\nabla_X \mathbf{N})$ coincides with the second fundamental form $\mathrm{II} : T(M^2) \to T(M^2)$ of the submanifold $M^2 \hookrightarrow Y^3$. Since II is symmetric the Clifford product $e_1 \cdot \mathrm{II}(e_1) + e_2 \cdot \mathrm{II}(e_2)$ is a scalar and equals $(-2H)$ where H denotes the mean curvature of the surface M^2. The Dirac operator D of M^2 defined by the formula

$$D(\varphi) = e_1 \cdot \nabla^{M^2}_{e_1} \varphi + e_2 \cdot \nabla^{M^2}_{e_2} \varphi$$

can now be expressed by the covariant derivative ∇^{Y^3} and the mean curvature vector:

$$e_1 \cdot \nabla^{Y^3}_{e_1}(\Phi) + e_2 \cdot \nabla^{Y^3}_{e_2}(\Phi) = D(\varphi) + H \cdot \mathbf{N} \cdot \varphi.$$

Suppose that the spinor field Φ on Y^3 is a real Killing spinor, i.e. there exists a number $\lambda \in \mathbb{R}^1$ such that for any tangent vector $\mathbf{T} \in T(Y^3)$ the derivative of Φ in the direction of $\mathbf{T}$ is given by the Clifford multiplication:

$$\nabla^{Y^3}_{\mathbf{T}}(\Phi) = \lambda \cdot \mathbf{T} \cdot \Phi.$$

For the restriction $\varphi = \Phi_{|M^2}$ we obtain immediately the equation

$$D(\varphi) = -2\lambda\varphi - H \cdot \mathbf{N} \cdot \varphi.$$

Using the decomposition $\varphi = \varphi^+ + \varphi^-$ the last equation is equivalent to the pair of equations

$$D(\varphi^+) = (-2\lambda - \mathrm{i}H)\varphi^-, \qquad D(\varphi^-) = (-2\lambda + \mathrm{i}H)\varphi^+.$$

We discuss two special cases.

Proposition 1. *Let M^2 be a minimal surface in Y^3. Then the restriction $\varphi = \Phi_{|M^2}$ of any real Killing spinor Φ on Y^3 is an eigenspinor of constant length on the surface M^2:*

$$D(\varphi) = -2\lambda\varphi.$$

On the other hand, suppose that Φ is a parallel spinor ($\lambda = 0$) on Y^3. Then we obtain

$$D(\varphi^+) = -\mathrm{i}H\varphi^-, \quad D(\varphi^-) = \mathrm{i}H\varphi^+.$$

If we introduce the spinor field $\varphi^* = \varphi^+ - \mathrm{i}\varphi^-$, a simple calculation shows

$$D(\varphi^*) = H\varphi^*.$$

The spinor field φ^* is given by

$$\begin{aligned}\varphi^* &= \varphi^+ - \mathrm{i}\varphi^- = \tfrac{1}{2}(\varphi + \mathrm{i} \cdot \mathbf{N} \cdot \varphi) - \tfrac{1}{2}\mathrm{i}(\varphi - \mathrm{i} \cdot \mathbf{N} \cdot \varphi)\\ &= \tfrac{1}{2}(1 - \mathrm{i})\varphi + \tfrac{1}{2}(-1 + \mathrm{i}) \cdot \mathbf{N} \cdot \varphi.\end{aligned}$$

Moreover, the length of φ^* is constant. This construction yields the following result.

Proposition 2. *Let Φ be a parallel spinor field defined on the 3-manifold Y^3 and denote by $\varphi = \Phi_{|M^2}$ its restriction to M^2. Define the spinor field φ^* on M^2 by the formula*

$$\varphi^* = \tfrac{1}{2}(1 - \mathrm{i})\varphi + \tfrac{1}{2}(-1 + \mathrm{i}) \cdot \mathbf{N} \cdot \varphi.$$

Then φ^* is a spinor field of constant length on M^2 satisfying the Dirac equation

$$D(\varphi^*) = H\varphi^*$$

where H denotes the mean curvature.

Remark 3. The map $\Phi \longmapsto \varphi^*$ associating to any parallel spinor Φ on Y^3 a solution of the equation $D(\varphi^*) = H\varphi^*$ is injective.

Remark 4. We can apply the above-mentioned formulas not only for Killing spinors. Indeed, for any spinor field Φ we have

$$D_{Y^3}(\Phi) = D(\varphi) + H \cdot \mathbf{N} \cdot \varphi + \mathbf{N} \cdot (\nabla^{Y^3}_{\mathbf{N}} \Phi),$$

where D_{Y^3} is the Dirac operator of the 3-manifold Y^3. Suppose there exists a function $\kappa : M^2 \to \mathbb{C}$ such that the normal derivative $(\nabla^{Y^3}_{\mathbf{N}}\Phi)$ of the spinor field Φ is described by κ:

$$(\nabla^{Y^3}_{\mathbf{N}}\Phi) = \kappa\Phi.$$

Then we obtain

$$D_{Y^3}(\Phi) = D(\varphi) + (H + \kappa)\mathbf{N}\cdot\varphi.$$

This formula (in arbitrary dimension) has been used for the calculation of the spectrum of the Dirac operator on hypersurfaces of the Euclidean space (see [1,13,14]).

3. Solutions of the Dirac equation with potential on Riemannian surfaces

Let (M^2, g) be an oriented, 2-dimensional Riemannian manifold with spin structure. $H : M^2 \to \mathbb{R}^1$ denotes a given smooth, real-valued function defined on the surface. In this part we study spinor fields φ on M^2 that are solutions of the differential equation

$$D(\varphi) = H\varphi.$$

If we decompose the spinor field into $\varphi = \varphi^+ + \varphi^-$ according to the splitting $S = S^+ \oplus S^-$ of the spinor bundle the equation we want to study is equivalent to the system

$$D(\varphi^+) = H\varphi^-, \qquad D(\varphi^-) = H\varphi^+.$$

To any solution φ of this equation we associate two forms $F_\pm$ defined for pairs $X, Y \in T(M^2)$ of tangent vectors:

$$F_+(X,Y) = \mathrm{Re}(\nabla_X\varphi^+, Y\cdot\varphi^-) \quad , \quad F_-(X,Y) = \mathrm{Re}(\nabla_X\varphi^-, Y\cdot\varphi^+).$$

Proposition 5.
(a) *$F_\pm$ are symmetric bilinear forms on $T(M^2)$.*
(b) *The trace of $F_\pm$ is given by $Tr(F_\pm) = -H|\varphi^\mp|^2$.*

Proof. The symmetry of $F_\pm$ is a consequence of the Dirac equation as well as the assumption that H is a real-valued function. Indeed, we have

$$\begin{aligned}
\mathrm{Re}(\nabla_{e_1}\varphi^+, e_2\varphi^-) &= \mathrm{Re}(e_1\cdot\nabla_{e_1}\varphi^+, e_1\cdot e_2\cdot\varphi^-)\\
&= \mathrm{Re}(H\varphi^- - e_2\cdot\nabla_{e_2}\varphi^+, e_1\cdot e_2\cdot\varphi^-)\\
&= H\cdot\mathrm{Re}(\varphi^-, e_1\cdot e_2\cdot\varphi^-) + \mathrm{Re}(\nabla_{e_2}\varphi^+, e_2\cdot e_1\cdot e_2\cdot\varphi^-)\\
&= 0 + \mathrm{Re}(\nabla_{e_2}\varphi^+, e_1\cdot\varphi^-).
\end{aligned}$$

Moreover, we calculate the trace of $F_\pm$:

$$\begin{aligned}
\mathrm{Tr}(F_\pm) &= \mathrm{Re}(\nabla_{e_1}\varphi^\pm, e_1\cdot\varphi^\mp) + \mathrm{Re}(\nabla_{e_2}\varphi^\pm, e_2\cdot\varphi^\mp)\\
&= -\mathrm{Re}(D(\varphi^\pm), \varphi^\mp) = -H|\varphi^\mp|^2. \qquad \square
\end{aligned}$$

We study now special solutions of the equation $D(\varphi) = H\varphi$, i.e. solutions with *constant length* $|\varphi| \equiv \mathrm{const} \neq 0$. It may happen that the components $\varphi^{\pm}$ have a non-empty zero set.

Proposition 6. *Suppose that the spinor field φ defined on the Riemannian surface M^2 is a solution of the equation*

$$D(\varphi) = H\varphi \quad \textit{with} \quad |\varphi| \equiv \mathrm{const} \neq 0.$$

Then the forms $F_{\pm}$ are related by the equation

$$|\varphi^+|^2 F_+ = |\varphi^-|^2 F_-.$$

Proof. In case one of the spinors φ^+ or φ^- vanishes at a fixed point $m_0 \in M^2$ the relation between F_+ and F_- is trivial. Otherwise there exists a neighbourhood V of the point $m_0 \in M^2$ such that both spinors φ^+ and φ^- are not zero at any point $m \in V$. The spinors

$$\frac{e_1 \cdot \varphi^-}{|\varphi^-|}, \qquad \frac{e_2 \cdot \varphi^-}{|\varphi^-|}$$

are an orthonormal base in S^+ with respect to the Euclidean scalar product $\mathrm{Re}(\cdot,\cdot)$. Therefore we obtain (on V)

$$\begin{aligned}\nabla_X \varphi^+ &= \mathrm{Re}\left(\nabla_X\varphi^+, \frac{e_1 \cdot \varphi^-}{|\varphi^-|}\right)\frac{e_1 \cdot \varphi^-}{|\varphi^-|} + \mathrm{Re}\left(\nabla_X\varphi^+, \frac{e_2 \cdot \varphi^-}{|\varphi^-|}\right)\frac{e_2 \cdot \varphi^-}{|\varphi^-|}\\ &= \frac{1}{|\varphi^-|^2}\{F_+(X,e_1)e_1 + F_+(X,e_2)e_2\} \cdot \varphi^-.\end{aligned}$$

A similar calculation yields the formula

$$\nabla_X \varphi^- = \frac{1}{|\varphi^+|^2}\{F_-(X,e_1)e_1 + F_-(X,e_2)e_2\} \cdot \varphi^+.$$

We multiply the equations by φ^+ and φ^-, respectively, and sum up. Then we obtain

$$\tfrac{1}{2}X(|\varphi^+|^2 + |\varphi^-|^2) = \mathrm{Re}(A(X)\varphi^-, \varphi^+),$$

where the endomorphism $A : T(M^2) \to T(M^2)$ is defined by

$$A(X) = \left\{\frac{F_+(X,e_1)}{|\varphi^-|^2} - \frac{F_-(X,e_1)}{|\varphi^+|^2}\right\}e_1 + \left\{\frac{F_+(X,e_2)}{|\varphi^-|^2} - \frac{F_-(X,e_2)}{|\varphi^+|^2}\right\}e_2.$$

Since $F_{\pm}$ are symmetric tensors, the endomorphism A is symmetric too. Moreover, the trace of A vanishes:

$$\mathrm{Tr}\,A = \frac{1}{|\varphi^-|^2}\mathrm{Tr}(F_+) - \frac{1}{|\varphi^+|^2}\mathrm{Tr}(F_-) = -H + H = 0.$$

The length of the spinor field φ is constant. This implies

$$\mathrm{Re}(A(X) \cdot \varphi^-, \varphi^+) = 0.$$

At any point $m \in V$ of the set V the spinors φ^+, φ^- are non-trivial. Then the rank of the endomorphisms $A : T(M^2) \to T(M^2)$ is not greater than 1. All in all, A is symmetric, $\mathrm{Tr}(A) = 0$ and $rg(A) \leq 1$, i.e. $A \equiv 0$. □

We now consider the sum

$$F = F_+ + F_-.$$

At points with $\varphi^+ \neq 0$ (or $\varphi^- \neq 0$) we have

$$\frac{F}{|\varphi|^2} = \frac{F_+ + F_-}{|\varphi^+|^2 + |\varphi^-|^2} = \frac{(|\varphi^-|^2/|\varphi^+|^2 + 1)F_-}{|\varphi^+|^2 + |\varphi^-|^2} = \frac{F_-}{|\varphi^+|^2}$$

as well as

$$\frac{F}{|\varphi|^2} = \frac{F_+ + F_-}{|\varphi^+|^2 + |\varphi^-|^2} = \frac{(1 + |\varphi^+|^2/|\varphi^-|^2)F_+}{|\varphi^+|^2 + |\varphi^-|^2} = \frac{F_+}{|\varphi^-|^2}.$$

The endomorphism $E : T(M^2) \to T(M^2)$ given by $g(E(X), Y) = F(X, Y)/|\varphi|^2$ is defined at all points of M^2 and the formulas derived in the proof of Proposition 6 in fact prove the following.

Proposition 7. *Let φ be a solution of the differential equation $D(\varphi) = H\varphi$ on a Riemannian surface (M^2, g) with a real-valued function $H : M^2 \to \mathbb{R}^1$. Suppose that the length $|\varphi| \equiv const \neq 0$ of the spinor field φ is constant. Then*

$$g(E(X), Y) = \frac{1}{|\varphi|^2}\mathrm{Re}(\nabla_X\varphi, Y \cdot \varphi)$$

defines a symmetric endomorphism $E : T(M^2) \to T(M^2)$ such that
(a) $\nabla_X\varphi^+ = E(X) \cdot \varphi^-$, $\nabla_X\varphi^- = E(X) \cdot \varphi^+$
(b) $Tr(E) = -H$.

For a given triple (M^2, g, E) of a Riemannian surface and symmetric endomorphism the existence of a non-trivial solution φ of the equation

$$\nabla_X\varphi = E(X) \cdot \varphi$$

implies certain integrability conditions. It turns out that in this way we obtain precisely the well-known Gauß and Codazzi equations of the classical theory of surfaces in Euclidean 3-space.

Proposition 8. *Let (M^2, g) be a 2-dimensional Riemannian surface with a fixed spin structure and suppose that $E : T(M^2) \to T(M^2)$ is a symmetric endomorphism. If there exists a non-trivial solution of the equation*

$$\nabla_X\varphi = E(X) \cdot \varphi, \quad X \in T(M^2)$$

then

(a) *(Codazzi equation):* $\nabla_X(E(Y)) - \nabla_Y(E(X)) - E([X, Y]) = 0$.

(b) *(Gauß equation):* $\det(E) = \frac{1}{4}G$, *where G is the Gaussian curvature of (M^2, g).*

Proof. We prove the two equations in a way similar to the derivation of the integrability conditions for the Riemannian metric in case the space admits a Killing spinor (see [2]). We differentiate the equation

$$\nabla_X\varphi = E(X)\cdot\varphi$$

and then we calculate the curvature tensor R^S of the spinor bundle S:

$$\begin{aligned} R^S(X, Y)\varphi &= \nabla_X\nabla_Y\varphi - \nabla_Y\nabla_X\varphi - \nabla_{[X,Y]}\varphi \\ &= \{\nabla_X(E(Y)) - \nabla_Y(E(X)) - E([X, Y]) + E(Y)E(X) \\ &\quad - E(X)E(Y)\}\cdot\varphi. \end{aligned}$$

On the other side, the curvature tensor $R^S : S \to S$ is given by the formula

$$R^S(e_1, e_2) = \tfrac{1}{2}R_{1212}\, e_1\cdot e_2.$$

Denote by $A(X, Y)$ the differential of E:

$$A(X, Y) = \nabla_X(E(Y)) - \nabla_Y(E(X)) - E([X, Y]).$$

A simple algebraic calculation in the spin representation then leads to the equations

$$\begin{aligned} -A(e_1, e_2)\varphi^- &= \left(2\det(E) + \frac{R_{1212}}{2}\right)\mathrm{i}\varphi^+ A(e_1, e_2)\varphi^+ \\ &= \left(2\det(E) + \frac{R_{1212}}{2}\right)\mathrm{i}\varphi^-, \end{aligned}$$

where $\{e_1, e_2\}$ form an orthonormal basis consisting of eigenvectors of E.

We multiply the first equation once by the vector $A(X; Y)$:

$$\|A(e_1, e_2)\|^2\varphi^- = -\left(2\det(E) + \frac{R_{1212}}{2}\right)^2\varphi^-$$

and then we conclude $A(X, Y) \equiv 0$ (Codazzi equation) as well as $\det(E) = -\frac{1}{4}R_{1212} = \frac{1}{4}G$ (Gauß equation). □

For a given triple (M^2, g, E) consisting of a Riemannian spin surface (M^2, g) and of a symmetric endomorphism E we will denote by $\mathcal{K}(M^2, g, E)$ the space of all spinor fields φ satisfying the equation $\nabla_X\varphi = E(X)\cdot\varphi$. It is invariant under the quaternionic structure $\alpha : S \to S$, i.e. $\mathcal{K}(M^2, g, E)$ is a quaternionic vector space (see Section 4). Denote by $(-H)$ the trace of E,

$$\mathrm{Tr}(E) = -H.$$

Then we have

$$\mathcal{K}(M^2, g, E) \subset \ker(D - H).$$

In this part of the paper we proved that any spinor field $\varphi \in \ker(D - H)$ of constant length belongs to one of the subspaces $\mathcal{K}(M^2, g, E)$ for a suitable symmetric endomorphism E, $Tr(E) = -H$.

Finally, we consider the lengths

$$L_+ = \|\varphi^+\|^2, \quad L_- = \|\varphi^-\|^2$$

of a non-trivial solution $\varphi \in \mathcal{K}(M^2, g, E)$. Using the integrability condition $\det(E) = \frac{1}{4}G$ (i.e. $\|E\|^2 = H^2 - \frac{1}{2}G$) as well as the well-known formula $D^2 = \Delta + \frac{1}{2}G$ for the square D^2 of the Dirac operator we can derive formulas for $\Delta(L_\pm)$:

$$\begin{aligned}\Delta(L_\pm) &= 2(\Delta(\varphi^\pm), \varphi^\pm) - 2\langle\nabla(\varphi^\pm), \nabla(\varphi^\pm)\rangle \\ &= 2(D^2(\varphi^\pm), \varphi^\pm) - 2\left(\frac{G}{2}\right) \cdot \|\varphi^\pm\|^2 - 2\|E\|^2\|\varphi^\mp\|^2 \\ &= 2\left(H^2 - \frac{G}{2}\right)(L_\pm - L_\mp) + 2\mathrm{e}(\mathrm{grad}(H) \cdot \varphi^\mp, \varphi^\pm)\end{aligned}$$

In particular, if $H \equiv \mathrm{const}$ is constant, the difference $u = L_+ - L_-$ satisfies the differential equation

$$\Delta(u) = 4\left(H^2 - \frac{G}{2}\right)u.$$

4. The period form of a spinor with $\nabla_X\varphi = E(X) \cdot \varphi$

We consider a spinor field φ on a Riemannian surface (M^2, g) such that

$$\nabla_X\varphi = E(X) \cdot \varphi$$

for a fixed symmetric endomorphism E. The spinor bundle S carries a quaternionic structure $\alpha : S \to S$ commuting with Clifford multiplication and interchanging the decomposition $S = S^+ \oplus S^-$ (see [3]). For any spinor field $\varphi = \varphi^+ + \varphi^-$ we define three 1-forms by

$$\xi^\varphi(X) = 2(X \cdot \varphi^+, \varphi^-),$$
$$\xi^\varphi_+(X) = (X \cdot \varphi^+, \alpha(\varphi^+)), \qquad \xi^\varphi_-(X) = (X \cdot \varphi^-, \alpha(\varphi^-)).$$

ξ^φ and ξ^φ_+ are $\Lambda^{1,0}$-forms, ξ^φ_- is a $\Lambda^{0,1}$-form. Indeed, $e_1 \cdot e_2$ acts on S^+ (on S^-) by multiplication by $(-\mathrm{i})$ (by i). Now we obtain

$$(\star\xi^\varphi)(e_1) = -\xi^\varphi(e_2) = 2(-e_2 \cdot \varphi^+, \varphi^-) = (-\mathrm{i}e_2 \cdot e_1 \cdot e_2 \cdot \varphi^+, \varphi^-) = -\mathrm{i}\xi^\varphi(e_1),$$

i.e. $\star\xi^{\varphi} = -\mathrm{i}\xi^{\varphi}$ holds. A similar calculation gives $\star\xi^{\varphi}_{+} = -\mathrm{i}\xi^{\varphi}_{+}$ and $\star\xi^{\varphi}_{-} = \mathrm{i}\xi^{\varphi}_{-}$. We split the 1-form ξ^{φ} into its real and imaginary part:

$$\xi^{\varphi} = w^{\varphi} + \mathrm{i}\mu^{\varphi}.$$

Moreover, we introduce the 1-form Ω^{φ}

$$\Omega^{\varphi} = \xi^{\varphi}_{+} - \xi^{\varphi}_{-}.$$

Then we have:

Proposition 9. *Let (M^2, g) be a Riemannian spin surface and $E : T(M^2) \to T(M^2)$ a symmetric endomorphism of trace $-H$. Suppose the spinor field φ is a solution of the equation $\nabla_X\varphi = E(X)\cdot\varphi$. Then*
(a) $\mathrm{d}w^{\varphi} = 0$.
(b) $\mathrm{d}\mu^{\varphi} = 2H\{|\varphi^-|^2 - |\varphi^+|^2\}\,\mathrm{d}M^2$.
(c) $\mathrm{d}\Omega^{\varphi} = 0$.

Proof. We calculate $\mathrm{d}w^{\varphi}$:

$$\begin{aligned}\tfrac{1}{2}\,\mathrm{d}w^{\varphi}(X,Y) &= X(\mathrm{Re}(Y\cdot\varphi^+,\varphi^-)) - Y(\mathrm{Re}(X\cdot\varphi^+,\varphi^-)) - \mathrm{Re}([X,Y]\cdot\varphi^+,\varphi^-)\\ &= \{g(X,E(Y)) - g(Y,E(X))\}|\varphi^-|^2\\ &\quad + \{g(X,E(Y)) - g(Y,E(X))\}|\varphi^+|^2.\end{aligned}$$

Since E is symmetric, we obtain $\mathrm{d}w^{\varphi} = 0$. A similar calculation shows the formula for $\mathrm{d}\mu^{\varphi}$. For the proof of $\mathrm{d}\Omega^{\varphi} = 0$ we first remark that the quaternionic structure $\alpha : S \to S$ and the hermitian product $(\cdot,\cdot)$ on S are related by

$$(\varphi_1, \alpha(\varphi_2)) = -\overline{(\alpha(\varphi_1), \varphi_2)}.$$

Using this formula we can transform $\mathrm{d}\xi^{\varphi}_{-}$ in the following way:

$$\begin{aligned}\mathrm{d}\xi^{\varphi}_{-}(X,Y) &= (Y\cdot E(X)\cdot\varphi^+, \alpha(\varphi^-)) + (Y\cdot\varphi^-, \alpha(E(X)\cdot\varphi^+))\\ &\quad -(X\cdot E(Y)\cdot\varphi^+, \alpha(\varphi^-)) - (X\cdot\varphi^-, \alpha(E(Y)\cdot\varphi^+))\\ &= -\overline{(\alpha(Y\cdot E(X)\cdot\varphi^+), \varphi^-)} - (E(X)\cdot Y\cdot\varphi^-, \alpha(\varphi^+))\\ &\quad +\overline{(\alpha(X\cdot E(Y)\cdot\varphi^+), \varphi^-)} + (E(Y)\cdot X\cdot\varphi^-, \alpha(\varphi^+))\\ &= -(E(X)\cdot Y\cdot\varphi^-, \alpha(\varphi^+)) - (E(X)\cdot Y\cdot\varphi^-, \alpha(\varphi^+))\\ &\quad +(E(Y)\cdot X\cdot\varphi^-, \alpha(\varphi^+)) + (E(Y)\cdot X\cdot\varphi^-, \alpha(\varphi^+)).\end{aligned}$$

On the other hand we calculate $\mathrm{d}\xi^{\varphi}_{+}$:

$$\begin{aligned}\mathrm{d}\xi^{\varphi}_{+}(X,Y) &= (Y\cdot E(X)\cdot\varphi^-, \alpha(\varphi^+)) + (Y\cdot\varphi^+, \alpha(E(X)\cdot\varphi^-))\\ &\quad -(X\cdot E(Y)\cdot\varphi^-, \alpha(\varphi^+)) - (X\cdot\varphi^+, \alpha(E(Y)\cdot\varphi^-))\\ &= (Y\cdot E(X)\cdot\varphi^-, \alpha(\varphi^+)) - (E(X)\cdot Y\cdot\varphi^+, \alpha(\varphi^-))\\ &\quad -(X\cdot E(Y)\cdot\varphi^-, \alpha(\varphi^+)) + (E(Y)\cdot X\cdot\varphi^+, \alpha(\varphi^-)).\end{aligned}$$

Finally we obtain

$$\begin{aligned}\mathrm{d}(\xi_-^\varphi - \xi_+^\varphi)(X,Y) &= -(\{E(X)\cdot Y + Y\cdot E(X)\}\varphi^-, \alpha(\varphi^+))\\ &\quad +(\{E(Y)\cdot X + X\cdot E(Y)\}\varphi^-, \alpha(\varphi^+))\\ &= 2\{g(E(X),Y) - g(E(Y),X)\}(\varphi^-, \alpha(\varphi^+))\end{aligned}$$

and $d(\xi_-^\varphi - \xi_+^\varphi) = 0$ follows again by the symmetry of E. □

Let us consider the case that (M^2, g) is isometrically immersed into the Euclidean space $\mathbb{R}^3$, Φ is a parallel spinor on $\mathbb{R}^3$ and the spinor field φ^* on M^2 defined by the formula

$$\varphi^* = \tfrac{1}{2}(\Phi_{|M^2} + \mathrm{i}\cdot\mathbf{N}\cdot\Phi_{|M^2}) + \tfrac{1}{2}\mathrm{i}(\mathrm{i}\cdot\mathbf{N}\cdot\Phi_{|M^2} - \Phi_{|M^2})$$

(see Section 2). In this case the forms w^{φ^*} and Ω^{φ^*} are given by the expressions

$$w^{\varphi^*}(X) = -\mathrm{Im}(X\cdot\Phi,\Phi), \quad \Omega^{\varphi^*}(X) = (X\cdot\Phi, \alpha(\Phi)),$$

and are exact 1-forms. Indeed, we defined functions $f:\mathbb{R}^3\to\mathbb{R}^1$ and $g:\mathbb{R}^3\to\mathbb{C}$ by

$$f(m) = -\mathrm{Im}\langle m\cdot\Phi,\Phi\rangle, \quad g(m) = \langle m\cdot\Phi, \alpha(\Phi)\rangle$$

and then we have $\mathrm{d}f = w^{\varphi^*}$, $\mathrm{d}g = \Omega^{\varphi^*}$. We remark that f and g describe in fact the isometric immersion $M^2 \hookrightarrow \mathbb{R}^3$ we started with. The 3-dimensional spinor $\Phi\in\Delta_3$ defines a real 3-dimensional subspace $\Delta_3(\Phi)$ by

$$\Delta_3(\Phi) = \{\Psi\in\Delta_3 : \mathrm{Re}(\Psi,\Phi) = 0\}.$$

The map $\Psi \to (-\mathrm{Im}(\Psi,\Phi), (\Psi,\alpha(\Phi)))$ is an isometry between $\Delta_3(\Phi)$ and $\mathbb{R}^1\oplus\mathbb{C} = \mathbb{R}^3$. Clearly, the immersion $M^2\hookrightarrow\mathbb{R}^3$ is given by

$$M^2 \ni m \longmapsto m\cdot\Phi \in \Delta_3(\Phi),$$

i.e. by the functions $f_{|M^2}$ and $g_{|M^2}$. With respect to $\mathrm{d}(f_{|M^2}) = w^{\varphi^*}$ and $\mathrm{d}(g_{|M^2}) = \Omega^{\varphi^*}$ we obtain a formula for the isometric immersion $M^2\hookrightarrow\mathbb{R}^3$:

$$\oint (w^{\varphi^*}, \Omega^{\varphi^*}) : M^2 \to \mathbb{R}^3.$$

(Weierstraß representation of the surface.)

In general, we call a solution φ of the differential equation $\nabla_X\varphi = E(X)\cdot\varphi$ *exact* iff the corresponding forms w^φ, Ω^φ are exact 1-forms. Using the definition

$$g(\mathrm{Hess}\,(h)(X),Y) = \tfrac{1}{2}\{g(\nabla_X(\mathrm{grad}(h)),Y) + g(X,\nabla_Y(\mathrm{grad}(h)))\}$$

of the Hessian of a smooth function h defined on a Riemannian manifold we obtain the following result.

Proposition 10. *Let $\varphi\in\mathcal{K}(M^2,g,E)$ be an exact solution of the differential equations $\nabla_X\varphi = E(X)\cdot\varphi$ with $\mathrm{d}f = w^\varphi$, $\mathrm{d}g = \Omega^\varphi$. Then*
(a) $\mathrm{Hess}(f) = 2\left(|\varphi^+|^2 - |\varphi^-|^2\right)E$.

(b) $|\text{grad}\, f|^2 = 4|\varphi^+|^2|\varphi^-|^2$.
(c) $\text{Hess}(g) = -4(\varphi^-, \alpha(\varphi^+))E$.
(d) $|\text{grad}(g)|^2 = (|\varphi^+|^2 - |\varphi^-|^2)^2$. □

In particular, the determinant of the Hessian of the function f is given by

$$\det(\text{Hess}\,(f)) = 4(|\varphi^+|^2 - |\varphi^-|^2)^2 \det(E) = (|\varphi^+|^2 - |\varphi^-|^2)^2 G.$$

Here we used Proposition 8, i.e. $\det(E) = \frac{1}{4}G$.

Corollary 11. *Let M^2 be a compact Riemannian spin-manifold and suppose that $\varphi \in \mathcal{K}(M^2, g, E)$ is an exact, non-trivial solution. Then the spinors φ^+ or φ^- vanish at least at one point. Moreover, there exists $m_0 \in M^2$ such that $G(m_0) \geq 0$.*

Proof. At a maximum point $m_0 \in M^2$ of f we have

$$\text{grad}(f)(m_0) = 0, \quad \det(\text{Hess}(f)(m_0)) \geq 0. \qquad \square$$

Recall that for any 2-dimensional Riemannian manifold (M^2, g) and any function $h : M^2 \to \mathbb{R}^1$ the 2-form

$$\{2\det(\text{Hess}\,(h)) - |\text{grad}(h)|^2 G\}\, \mathrm{d}M^2 = \mathrm{d}\mu^1$$

is exact (see [9, p. 47]). Using this formula for $h = f$ in case of an exact solution $\varphi \in \mathcal{K}(M^2, g, E)$ we obtain

$$\int_{M^2} (|\varphi^+|^2 - |\varphi^-|^2)^2 G = 2\int_{M^2} |\varphi^+|^2|\varphi^-|^2 G.$$

Corollary 12. *Let M^2 be a compact Riemannian spin manifold and suppose that $\varphi \in \mathcal{K}(M^2, g, E)$ is an exact solution. Then*

$$\int_{M^2} (|\varphi|^4 - 6|\varphi^+|^2|\varphi^-|^2)G = 0.$$

We again discuss the last formula in case of an isometrically immersed surface $M^2 \hookrightarrow \mathbb{R}^3$ and a given parallel spinor Φ on $\mathbb{R}^3$. We apply the integral formula to the spinor $\varphi^* = \varphi^*_+ + \varphi^*_-$ where

$$\varphi^*_+ = \tfrac{1}{2}(\Phi + \mathrm{i}\mathbf{N} \cdot \Phi), \qquad \varphi^*_- = \tfrac{1}{2}\mathrm{i}(-\Phi + \mathrm{i} \cdot \mathbf{N} \cdot \Phi).$$

In this case we have

$$|\varphi^*_+|^2 = \tfrac{1}{2}|\Phi|^2 + \tfrac{1}{2}\langle \mathrm{i}\mathbf{N} \cdot \Phi, \Phi\rangle, \quad |\varphi^*_-|^2 = \tfrac{1}{2}|\Phi|^2 - \tfrac{1}{2}\langle \mathrm{i}\mathbf{N} \cdot \Phi, \Phi\rangle$$

and ($|\Phi| \equiv 1$) therefore

$$1 - 6|\varphi^*_+|^2|\varphi^*_-|^2 = -\tfrac{1}{2} + \tfrac{3}{2}\langle \mathrm{i} \cdot \mathbf{N} \cdot \Phi, \Phi\rangle^2.$$

Consequently, the integral formula yields

$$\int_{M^2} G = 3 \int_{M^2} \langle \mathrm{i}\mathbf{N}\Phi, \Phi \rangle^2 G.$$

The spinors $\mathrm{i}\Phi$ as well as $\mathbf{N} \cdot \Phi$ belong to $\Delta_3(\Phi) \subset \Delta_3$, the space of the immersion $M^2 \hookrightarrow \mathbb{R}^3 = \Delta_3(\Phi)$. The last formula means therefore

$$\int_{M^2} G = 3 \int_{M^2} \langle \mathbf{N}, \alpha_3 \rangle^2 G$$

for the unit vector $\alpha_3 = \mathrm{i}\Phi \in \Delta_3(\Phi) = \mathbb{R}^3$.

5. The spin formulation of the theory of surfaces in $\mathbb{R}^3$

An oriented, immersed surface $M^2 \hookrightarrow \mathbb{R}^3$ inherits from $\mathbb{R}^3$ an inner metric g, a spin structure and a solution φ of the Dirac equation

$$D(\varphi) = H\varphi$$

of constant length $|\varphi| \equiv 1$ where H denotes the mean curvature of the surface. The spinor field φ on M^2 is the restriction of a parallel spinor field Φ of the Euclidean space $\mathbb{R}^3$. The period forms w^φ and Ω^φ are exact and the immersion $M^2 \hookrightarrow \mathbb{R}^3$ is given by integration of the $\mathbb{R}^1 \oplus \mathbb{C} = \mathbb{R}^3$ valued form $(w^\varphi, \Omega^\varphi)$. At least locally the converse is true: Given an oriented, 2-dimensional Riemannian manifold (M^2, g) with a fixed spin structure and a solution of constant length of the Dirac equation $D(\varphi) = H\varphi$ for some smooth function $H : M^2 \to \mathbb{R}^1$, there exists a symmetric endomorphism $E : T(M^2) \to T(M^2)$ such that $\varphi \in \mathcal{K}(M^2, g, E)$. Moreover, $2E$ is the second fundamental form of an isometric immersion $(M^2, g) \to \mathbb{R}^3$. We formulate this description of the theory of surfaces in $\mathbb{R}^3$ in the following

Theorem 13. *Let (M^2, g) be an oriented, 2-dimensional Riemannian manifold and $H : M^2 \to \mathbb{R}^1$ a smooth function. Then there is a correspondence between the following data:*
1. *An isometric immersion $(\tilde{M}^2, g) \to \mathbb{R}^3$ of the universal covering $\tilde{M}^2$ into the Euclidean space $\mathbb{R}^3$ with mean curvature H.*
2. *A solution φ with constant length $|\varphi| \equiv 1$ of the Dirac equation $D(\varphi) = H \cdot \varphi$.*
3. *A pair (φ, E) consisting of a symmetric endomorphism E such that $Tr(E) = -H$ and a spinor field φ satisfying the equation $\nabla_X \varphi = E(X) \cdot \varphi$.*

We apply now the well-known formulas for the change of the Dirac operator under a conformal change of the metric. Suppose that $\tilde{g} = \sigma g$ are two conformally equivalent metrics on M^2 where $\sigma : M^2 \to (0, \infty)$ is a positive function. Denote by D and $\tilde{D}$ the Dirac operator corresponding to the metric g and $\tilde{g}$, respectively. Then

$$\tilde{D}(\varphi) = \sigma^{-3/4} D(\sigma^{1/4}\varphi)$$

holds (see [2]). Let us consider a solution φ of the Dirac equation

$$D(\varphi) = \lambda\varphi$$

on (M^2, g) and suppose that φ never vanishes. We introduce the Riemannian metric $\tilde{g} = |\varphi|^4 g$ as well as the spinor field $\varphi^* = \varphi/|\varphi|$. Then we obtain

$$\tilde{D}(\varphi^*) = \frac{\lambda}{|\varphi|^2}\varphi^*, \quad |\varphi^*| \equiv 1,$$

and thus an isometric immersion $(\tilde{M}^2, |\varphi|^4 g) \to \mathbb{R}^3$ with mean curvature $H = \lambda/|\varphi|^2$.

Theorem 14. *Let (M^2, g) be an oriented, 2-dimensional Riemannian manifold. Any spinor field φ without zeros that is a solution of the equation*

$$D(\varphi) = \lambda\varphi$$

defines an isometric immersion $(\tilde{M}^2, |\varphi|^4 g) \hookrightarrow \mathbb{R}^3$ with mean curvature $H = \lambda/|\varphi|^2$.

Remark 15. Consider the case that $M^2 \hookrightarrow S^3$ is a minimal surface in S^3. Let Φ be a real Killing spinor on S^3, i.e.

$$\nabla_{\mathbf{T}}(\Phi) = \tfrac{1}{2}\mathbf{T} \cdot \Phi.$$

The restriction $\varphi = \Phi_{|M^2}$ is an eigenspinor of the Dirac operator on M^2 with constant length (Proposition 1). Therefore φ defines an isometric immersion of $(\tilde{M}^2, g) \hookrightarrow \mathbb{R}^3$ with mean curvature $H \equiv -1$. This transformation associates to any minimal surface $M^2 \hookrightarrow S^3$ a surface of constant mean curvature $H \equiv -1$ in $\mathbb{R}^3$, a well-known construction (see [6]).

Remark 16. Using the described correspondence between isometric immersions of surfaces into $\mathbb{R}^3$ and solutions of the Dirac equation $D(\varphi) = H \cdot \varphi$ one can immediately remark that several statements of the elementary theory of surfaces are equivalent to several statements concerning solutions of the twistor equation (see [2]). For example, in [7] (see also Proposition 8) one can find the following theorem: if $f : M^2 \to \mathbb{R}^1$ is a real-valued function such that the equation

$$\nabla_{\mathbf{T}}(\varphi) + \tfrac{1}{2} f \cdot \mathbf{T} \cdot \varphi = 0$$

admits a non-trivial solution then f is constant and $f^2 = G$. In the theory of surfaces this statement correspondends to the fact that an umbilic surface is a part of the sphere or the plane. Indeed, an umbilic surface $M^2 \hookrightarrow \mathbb{R}^3$ admits a spinor field φ such that

$$\nabla_{\mathbf{T}}(\varphi) + \tfrac{1}{2} H\mathbf{T} \cdot \varphi = 0$$

and therefore $H^2 = G = \text{const}$, i.e. the second fundamental form is proportional to the metric. In a similar way one can translate other facts of the theory of surfaces into properties of solutions of the equation $\nabla_X \varphi = E(X) \cdot \varphi$.

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